Time evolution of a crack

Question 1: How to define the time-evolution of a crack inside some material, from a mathematical point of view???

Question 2: Once a suitable notion of crack evolution has been identified, how to prove existence of such an evolution, during some given time interval $[0, T]$?

Question 1: suitable notion of evolution?

POLITECNICO DI TORINO
Dipartimento di Scienze Matematiche G.L. Lagrange

Observation: let's agree that a crack propagates when it is energetically "convenient"

Idea: consider the energy associated to the system and define a crack evolution as a minimizer of the energy

As a first step: define the crack evolution as a global minimizer of the energy associated to the system

Note: the technical mathematical definition of quasi-static crack evolution is based on the ideas above.

Technical: the concept of quasi-static crack evolution, based on Griffith's criterion, was first introduced in [Francfort-Marigo, 1998] and rigorously developed in [Dal Maso, Francfort, Toader, 2005]. For further references, see e.g., [Bourdin-Francfort-Marigo, 2008], [Mielke-Roubicek, 2015].

Question 2: existence of a crack evolution?

General strategy: time-discretization, that is

* consider a discrete subdivision of the time interval $[0, T]$
* define a suitable discrete-in-time version of the crack evolution
* pass to the limit as the time step goes to zero (from discrete to continuous) and prove that the limit evolution enters the definition of crack evolution

Observation: in general, even though there is a general strategy, the existence proof is not trivial and depends on the energy functional under investigation.

Further problem: global minimization ...?

Note: from a mathematical point of view it is natural that the first step is to consider global minimizers of the energy, but.... let's consider the following example:

consider a sheet of paper with an initial crack on one side and pull it in order to let the crack propagate. In general, nothing may happen for some time, and then the crack propagates till the end of the paper.

behavior of the global minimizer: the instant you start to pull... the paper is immediately broken into two parts (this is the configuration of the global minimizer)

... Next question: how to define a notion of crack evolution based on local minimizers of the energy???

Local minimizers?

$*$ Since in general there is non-uniqueness of minimizers, the main question is: which selection criterion shall we impose?

$*$ Idea: use vanishing viscosity. In a finite dimensional setting, this means: given an energy functional $f(t,x) : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, a quasi-static evolution $t \mapsto x(t)$ may be defined as a solution to

$$(P) \qquad \nabla_x f(t, x(t)) = 0$$

$*$ One option to obtain a solution of this problem, is to consider the viscous (more regular) problem

$$(P_\varepsilon) \qquad \varepsilon \dot{x}_\varepsilon(t) + \nabla_x f(t, x_\varepsilon(t)) = 0$$

and study the limit as $\varepsilon \to 0$, proving that (up to subsequences) $x_\varepsilon(t)$ converge to a solution $x(t)$ of the original problem (P) [Zanini, 2007]

Vanishing viscosity applied to crack evolution

POLITECNICO DI TORINO
Dipartimento di Scienze Matematiche G.L. Lagrange

NOTE: in order to apply vanishing viscosity to the problem of crack evolution, I assume that the crack path is known. This means that I am not interested in where the crack will appear in the material, but I am interested in the behavior of the crack propagation, that is, to model the smooth propagation of the crack, and, mainly, when the crack has a jump discontinuity (in time).

[The assumption that the crack path is prescribed allows to work in "simpler" function spaces - Sobolev spaces - in order to focus on the behavior of the crack propagation]

Vanishing viscosity approach

Vanishing viscosity approach was successfully applied to obtain existence results also for crack evolutionary models

A few examples

- crack propagation: [Toader-Zanini, 2009], [Cagnetti, 2008], [Knees-Mielke-Zanini, 2008,2010], [Lazzaroni-Toader, 2011], [Almi, 2016], [Artina-Cagnetti-Fornasier-Solombrino, 2016], [Crismale-Lazzaroni, 2017]
- damage: [Knees-Rossi-Zanini, 2013, 2015, 201?], [Knees-Negri, 2017]
- plasticity: [Dal Maso-DeSimone-Mora-Morini, 2008], [Dal Maso-DeSimone-Solombrino 2011,2012], [Babadjian, Francfort, Mora, 2012]
- abstract rate-independent evolutions: [Efendiev-Mielke, 2006],[Mielke-Rossi-Savaré 2009,2012], [Mielke-Zelik, 2013], [Mielke-Roubicek, 2015]

Some numerics

The general strategy to prove existence goes through time-discretization, which suggests the possibility to do some numerical implementation:

[Knees-Schroeder, 2013]: (see also references therein)

- numerical analysis in the case of propagation of a single crack in a linearly elastic material
- both notions of evolutions, based on global stability and via vanishing viscosity, are discussed and convergence of evolutions of fully discretized models (with respect to space and time) is proved
- the convergence proofs rely on regularity estimates for the elastic field close to the crack tip and local and global finite element error estimates
- the theoretical results are illustrated with some numerical calculations, providing a further proof that the two notions are different

see also [Knees-Negri, 2017]

Further steps: open problems

Further existence results via vanishing viscosity: for example [Knees-Zanini, in preparation]: the case of time-discontinuous loading

... new frontiere: dynamic fracture model: There is not a general formulation, but as an example we cite

[Dal Maso-Lazzaroni-Nardini, 2016] (see also references therein):